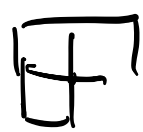
RSK "ROBINSON - SCHENSTED - KNUTH":

FIRST VERSION (ROBINSON):

S_n = SYMMETRIC GROUP.

A PARTITION $\lambda \vdash n$ IS ONE WITH $\sum \lambda_i = n$

A STANDARD TABLEAU OF SHAPE λ IS A FILLING OF THE YOUNG DIAGRAM BY INTEGERS $\{1, 2, \dots, n\}$ ROWS COLS STRICTLY INCREASE, EACH NUMBER APPEARS EXACTLY ONCE

SYT FOR  $\lambda = (2,1)$

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$$

THERE IS AN IRR. REP'N OF S_n PARAMETRIZED BY λ AND $\dim(\pi_\lambda^{S_n})$ = # OF SYT OF SHAPE λ.

THEOREM (RSK): THERE IS A BIJECTION S_n AND PAIRS OF SYT OF SAME SHAPE λ, $\lambda \vdash n$.

S_3 THREE SHAPES:

(3) $\quad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}$ $\quad$ DIM $\pi^{S_3}_{(3)} = 1$

TRIVIAL REP

$(1,1,1)$ $\quad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$ $\quad$ SIGN REP.

$(2,1)$ $\quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$ $\quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$ $\quad$ IRR. OF DEGREE 2.

REF: LIE GROUPS BOOK CH. 37 FOR PARAMETRIZATION OF REPS OF S_n BY PARTITIONS. BOOK OF SAGAN FOR A DIFFERENT APPROACH.

I WILL EXPLAIN THE ALGORITHM TODAY

THEOREM RSK: THERE IS A BIJECTION BETWEEN WORDS OF LENGTH n IN ALPHABET $\{1,2,\dots,n\}$ AND PAIRS (T_1,T_2) OF TABLEAUX OF SHAPE $\lambda \vdash k$, T_1 IS SSYT AND T_2 A SYT.

THEOREM RSK: THERE IS A BIJECTION BETWEEN $k \times n$ MATRICES IN $\mathbb{N} = \{0,1,2,\dots\}$ WITH PAIRS OF SSYT (T_1,T_2) OF SAME SHAPE
T_1 IN $\{1,2,\dots,k\}$
T_2 IN $\{1,2,\dots,n\}$.

SEE FULTON, YOUNG TABLEAUX FOR RSK AND CONNECTIONS WITH GEOMETRY.

CRYSTAL CONNECTION: THERE ARE DIFFERENT WAYS OF EMBEDDING $\mathcal{B}_\lambda$ CRYSTAL OF TABLEAUX IN $\mathcal{B} \otimes \dots \otimes \mathcal{B}$.

TO UNDER THIS SETTUP IDEAS OF KNUTH
LASCOUX, SCHÜTZENBERGER BECAME IMPORTANT.
CRYSTALS GIVE A GOOD FRAMEWORK FOR
THIS.

<u>SCHENSTED</u> <u>INSERTION</u>;

SUPPOSE WE HAVE A SSYT T (ROWS; WEAKLY INCREASE
AND WE HAVE ANOTHER COL; STRICTLY INCREASE)
ENTRY THAT WE WANT TO
ADD TO IT TO OBTAIN A TABLEAU T'.
(ADD ONE BOX TO YOUR DIAGRAM.)

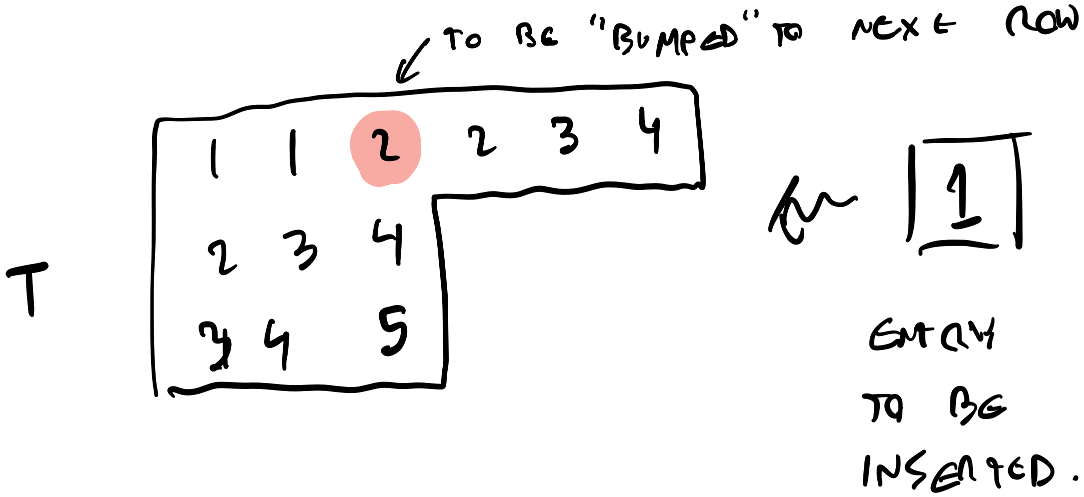
LET j BE THE ENTRY WE WANT
TO INSERT.

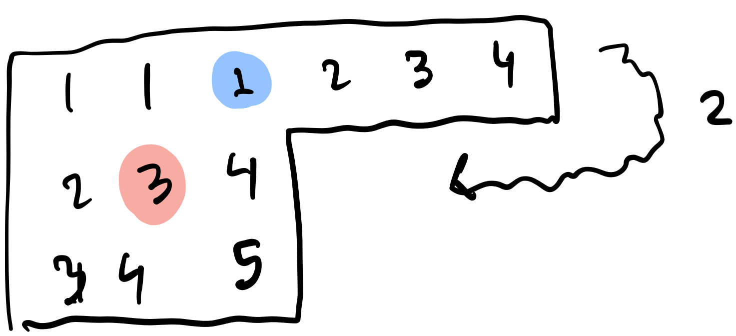
IF $j \geq$ FIRST ENTRY IN FIRST ROW,
PUT IT AFTER THAT ENTRY,
THEN STOP.

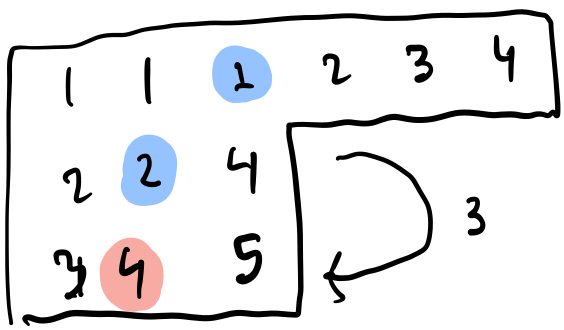
IF NOT, FIND THE LAST ENTRY THAT

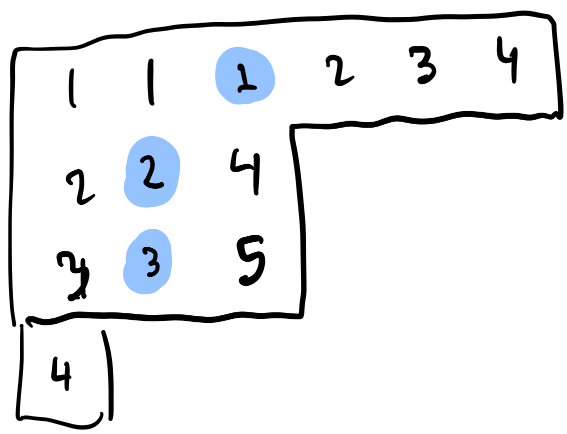
IS LARGER THAN j, REPLACE IT BY j, TAKE THE ENTRY THAT IS "BUMPED" AND INSERT IT INTO SECOND ROW.

CONTINUE.





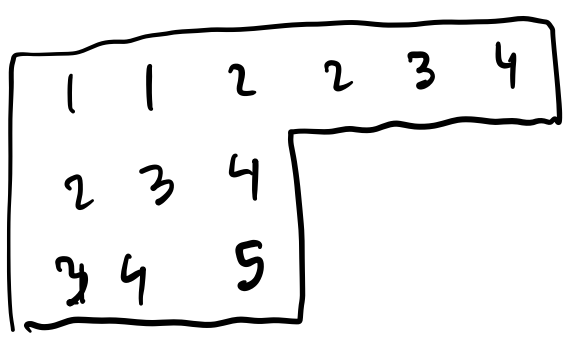




ANSWER

$T \leftarrow j$

ANOTHER EXAMPLE:



$\boxed{3}$

$$\begin{array}{cccccc}
1 & 1 & 2 & 2 & 3 & 4 \\
2 & 3 & 4 & \boxed{4} & & \\
3 & 4 & 5 & & &
\end{array}$$

SCHENSTED INSERTION.

RSK: BIJECTION BETWEEN WORDS

$i_1 i_2 \cdots i_k$ $\qquad$ $i_1, \dots, i_k \in \{1, 2, \dots, n\}$

IN ALPHABET $\{1, \dots, n\}$.

AND PAIRS OF TABLEAUX OF SAME SHAPE $\lambda \vdash k$. FIRST TABLEAU T_1 IS SSYT IN $\{1, 2, \dots, n\}$

AND T_2 IS SYT IN $\{1, 2, \dots, k\}$.

ALGORITHM: TO OBTAIN T_1, BUILD UP T_1 BY SUCCESSIVELY INSERTING

$i_1, i_2, \dots, i_k$. T_2 WILL BE DISCUSSED LATER.

EXAMPLE:

32131223

$\emptyset \leftarrow 3$

$\begin{array}{c} 3 \end{array} \leftarrow 2$

$\begin{array}{c} 2 \\ 3 \end{array} \leftarrow 1$

$\begin{array}{c} 1 \\ 2 \\ 3 \end{array} \leftarrow 3$

$\begin{array}{ll} 1 & 3 \\ 2 & \\ 3 & \end{array} \leftarrow 1$

$\begin{array}{ll} 1 & 1 \\ 2 & 3 \\ 3 & \end{array} \leftarrow 2$

$\begin{array}{lll} 1 & 1 & 2 \\ 2 & 3 & \\ 3 & & \end{array} \leftarrow 2$

$\begin{array}{llll} 1 & 1 & 2 & 2 \\ 2 & 3 & & \\ 3 & & & \end{array} \leftarrow 3$

$$T_1 = \begin{array}{lllll} 1 & 1 & 2 & 2 & 3 \\ 2 & 3 & & & \\ 3 & & & & \end{array}$$

NOT ENOUGH INFO TO RECONSTRUCT 32131223.

WE SUPPLEMENT T_1 BY A "RECORDING TABLEAU"

THE RECORDING TABLEAU HAS SAME SHAPE $\lambda = (5,2,1)$ IN EXAMPLE AND RECORDS LOCATION OF THE BOXES IN ORDER THEY WERE ADDED

$$T_2 = \begin{array}{|c|c|c|c|c|} \hline 1 & 4 & 6 & 7 & 8 \\ \hline 2 & 5 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}$$

"RECORDING TABLEAU".

THERE IS ENOUGH INFORMATION IN (T_1, T_2) TO RECONSTRUCT THE WORD.

COROLLARY: $n^k = \#$ WORDS

$$= \sum_{\lambda \vdash k} \#\left\{\begin{matrix}\text{SSYT OF} \\ \text{SHAPE } \lambda \\ \text{IN } 1,2,\dots,n\end{matrix}\right\} \#\left\{\begin{matrix}\text{SYT IN} \\ \{1,2,\dots,k\}\end{matrix}\right\}.$$

$$= \sum_{\lambda \vdash k} \left(\dim \pi_\lambda^{GL(n)}\right)\left(\dim \pi_\lambda^{S_k}\right).$$

INTERPRETATION:

$$\underbrace{\mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n}_{k \text{ TIMES}} = \bigoplus_{\lambda \vdash k} \pi_\lambda^{GL(n,\mathbb{C})} \otimes \pi_\lambda^{S_k}$$

AS $GL(n,\mathbb{C}) \times S_k$ MODULES.

SCHUR ~~WEYL~~ DUALITY.

[REF: MY LIE GROUPS BOOK]
CH. 34 - 39

IF $w = i_1, \dots, i_k$ IS A PERMUTATION MEANING $i_1, \dots, i_k$ ARE $1, 2, \dots, k$ IN SOME ORDER (AND $n = k$).

CLEARLY T_1 IS ALSO A SYT.

MAGIC: IF $\sigma \rightsquigarrow (T_1, T_2)$

$$\sigma^{-1} \rightsquigarrow (T_2, T_1)$$

NEXT TIME: RSK AND CRYSTALS.

ANOTHER REFERENCE

<u>KNUTH</u>: THE ART COMPUTER PROGRAMMING.

RSK $\rightsquigarrow$ SPANNING TREES, LABELED TREES.